Anisotropic spectra of acoustic type turbulence

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Abstract

We consider the problem of spectra for acoustic type of turbulence generated by shocks being randomly distributed in space. We show that for turbulence with a weak anisotropy such spectra have the same dependence in k-space as the Kadomtsev-Petviashvili (KP) spectrum: $E(k) \sim k^{-2}$. However, the frequency spectrum has always the falling $\sim \omega^{-2}$, independently on anisotropy. In the strong anisotropic case the energy distribution relative to wave vectors takes anisotropic dependence forming in the large k region the spectra of the jet type.

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1 Introduction

The acoustic type turbulence is ubiquitous in space and laboratory plasmas. Typical example represents MHD turbulence (for review see [1] and references therein) in the presence of external magnetic field at a moderate level of β (β is the ratio of kinetic plasma pressure to the magnetic pressure). When wave amplitudes are small the turbulence can be described as an ensemble of linear noninteracting waves with their frequencies ω_k and wave-vectors k. In the long-wave limit wave frequency can be considered as linear function of a wave number: $\omega_k \sim k$. In the thermodynamic equilibrium limit these waves are distributed according to the Jeans law so that the energy of each wave $\varepsilon_k = T$ where T stands for the "wave's" temperature. When the wave amplitudes increase (with the growth of the energy source) we first come to the regime of weak turbulence. The nonlinear effects are weak in comparison with linear wave dispersion but statistical characteristics of the system change significantly. In the weak turbulence regime, in addition to the thermodynamic distribution solution, it emerges the additional solution having the spectrum of the Kolmogorov type:

$$\varepsilon(k) = C\left(\rho P c_s\right)^{1/2} k^{-7/2}$$

where P is the energy flux toward small scales, ρ the density, c_s the sound velocity and C is the dimensionless constant of the order of one. In the spherical normalization this energy distribution reads as follows

$$E(k) = 4\pi k^2 \varepsilon_k = 4\pi C \left(\rho P c_s\right)^{1/2} k^{-3/2}. \tag{1}$$

This spectrum was found in 1971 by Zakharov and Sagdeev [2] as the exact solution of the wave kinetic equation for acoustic waves in isotropic medium, when in the long-wave region the dispersion relation is linear $\omega_k \approx c_s k$ (for details, see [3]). In this case the criterion of weak turbulence that is determined as the weakness of the nonlinearity with respect to the wave dispersion, is written as

$$\frac{\Delta\rho}{\rho} \ll k^2 \Lambda^2 \tag{2}$$

where $\Delta\rho$ is the fluctuating part of the density ρ, and Λ is the dispersion length [1]. For larger amplitudes when the nonlinear effects become comparable or larger the wave dispersion this criterion (2) breaks down. As it is well known in gas dynamics, in this case the main nonlinear effect for acoustic waves is nothing else but the wave breaking that results in the formation of shocks (or discontinuities). It is well known also that this process in compressible flows can be treated in terms of the formation of folds in the classical catastrophe theory [5]. In the gas-dynamic case, breaking areas can be completely characterized using the mapping defined by the transition from the usual Eulerian to the Lagrangian description. A vanishing of the Jacobian of the mapping corresponds to the emergence of a singularity for the spatial derivatives of the velocity and density of the fluid. Physical meaning of this effect corresponds to the intersection of particle trajectories. In the general situation first time the Jacobian vanishing happens in one isolated point. In collisionless plasma this process can continue and leads further to form the multi-flow region expanding in the transverse (relative to the main flow) direction according to the following scaling law $R_\perp \sim \sqrt{t_0 - t}$, where t_0 stands for the first moment when Jacobian J turns to zero, while the region width (in the longitudinal direction, along the main flow) increases more slowly, $R_\parallel \sim (t_0 - t)^{3/2}$ (see, e.g. [6, 7]). Thus, the result of the breaking consists in the formation of structures in the form of pancakes with very different characteristic spatial scales along the flow and in transverse directions. In optics quite similar structures are called caustics [8]. The simplest way to represent such structures is to consider them as disks (singular manifolds) on which the density undergoes jumps vanishing at the disk boundary. It is worth noting that in classical hydrodynamics these regions are considered as "forbidden" where the solution can not be constructed anymore in a well defined way, that means as a single value solution. There are special procedures of construction of so called "shock type" solution, see the book of Whitham [9] for more

[1] In the case of three-wave interacting waves of different branches the criterion of weak turbulence consists in smallness of the inverse nonlinear time defining by the kinetic equations in comparison with the maximal growth rate of the corresponding decay instability for the monochromatic wave. The latter in such system represents the inverse time of randomization (see, e.g. [4])

detail. However, in collisionless plasma physics the detailed description is written in terms of velocity distribution function that satisfies to Vlasov equation in the phase space $(\mathbf{p_i}, \mathbf{r_i})$. For this function the regions where the hydrodynamic solution becomes multi-valued and poorly determined are the areas where the distribution function undergoes the transition from the one single peak distribution to the one having three peaks, however there is no any crucial break in the phase space. From the other hand the spatial derivatives of some characteristics have infinite gradients also, but this effect can be explicitly described in the frame of the same Vlasov equation for collisionless plasmas. Formation of simple waves making use of such a description was considered by Gurevich and Pitaevskii [10].

In weak turbulence waves are assumed to be randomly distributed with a weak correlation between waves because of weak nonlinear interaction. The process of breaking is purely coherent and, respectively, the expanding caustics should be treated as coherent elementary entities of wave turbulence. We will call the turbulence where this effect becomes important a moderately strong turbulence if the density of caustics is still small enough to neglect their intersections that are supposed to be rare. The caustics can be described if their centers and orientations are determined. We will assume their distribution in space to be random. From the other hand, the jumps as the density singularities are known to result in a power-law tails in the short-wavelength part of the turbulence spectrum. This idea was first proposed by Phillips [11] and allowed him to determine the water-wave turbulence spectrum in the presence of whitecaps, i.e., of the singularities on the fluid surface. Later the very same idea was developed by Saffman [12] to determine the isotropic spectrum of two-dimensional hydrodynamic turbulence at high Reynolds numbers (for details see [13, 14]). Kadomtsev and Petviashvili (KP) [15] suggested that the acoustic turbulence can be considered as a randomly distributed set of shocks. For the isotropic case they found the spectrum of the energy distribution, now known as the KP spectrum,

$$E(k) \sim k^{-2}. \tag{3}$$

The goal of our paper is to determine how the KP spectrum is modified in the presence of strong enough anisotropy in plasma. Such an anisotropy can be entirely inherent, either due to the mean magnetic field or because of the anisotropy of turbulence source/pumping. It is worth noting that in the weak-turbulence regime, when the wave dispersion of acoustic waves can be neglected, the angular distribution of the spectrum repeats the anisotropy of the pumping because of three-wave resonant conditions, $\omega_k = \omega_{k_1} + \omega_{k_2}$ and $\mathbf{k} = \mathbf{k_1} + \mathbf{k_2}$. In this case the additional anisotropy of the spectra can appear either due to weak dispersion [16, 3] or because of nonlinear renormalizations to the wave kinetic equation [17]. For weak MHD turbulence, in both low and high beta plasmas, the situation is familiar to the weak acoustic turbulence in isotropic media for the nonlinear interacting fast magnetoacoustic waves which dispersion is mainly defined by modulus of the wave vector. However, at $\beta \ll 1$ slow magnetoacoustic waves undergo strong anisotropy in both dispersion and nonlinear interaction due to the external magnetic field. As it was shown in the paper [18] by one of the authors, it results in

appearance of anisotropic Kolmogorov-type spectra with power dependences relative to both longitudinal and transverse projections of the wave vector. Such character of the spectra can be established also for the interaction of Alfvenic and slow magnetoacoustic waves at low beta plasma [4]. For $\beta \gg 1$ slow magnetoacoustic waves have the same dispersion as the Alfvenic waves. This degeneracy also changes significantly the nonlinear interaction between waves and leads in the weak-turbulence regime to the spectrum of the Kraichnan-Iroshnikov type [19] (for details see [20, 1]).

In the present paper, for moderately strong acoustic turbulence, we study how the angular ordering of shocks can change the angular structure of the spectrum. We show that for the strong anisotropy the spectrum has the jet-type behavior with power increasing along the jet with the same exponent as for the isotropic KP spectrum and the falling-off dependence in transverse direction $\sim k_\perp^{-5}$. The latter originates from the contribution from boundaries of caustics (disks). It is necessary to mention that for two-dimensional acoustic turbulence the phenomenon of the jet-type spectra generated by shocks was observed first time in the numerical experiments [21]. However, comparably small spatial resolution could not allow the authors [21] to treat the fine structure of jets.

The second objective of our work is to show how the presence of the KP spectrum and its relative role can be evaluated making use of real experimental data (for instance, from spacecraft data for solar wind turbulence) in the presence of weak turbulence tails falling off more slowly than the KP spectrum. This question is very important because in the case when weak and strong turbulent components co-exist, the KP component of the spectrum can provide a quantitative measure of the relative role of coherent structures for moderate acoustic type turbulence. Hereafter we shall also show that the spectral index of the isotropic KP spectrum (3) as a function of the wave-vector does not depend on space dimension. It is worth noting that the Fourier transform of the density single point auto-correlation function dependence upon time shows the same power-law spectral index in time domain (in frequency) as in the spatial domain (spectrum in k) 3.

2 Isotropic spectra

Let us consider the contribution of shocks to the frequency spectrum, i.e., the energy distribution dependence on frequency. To find the spectrum one should calculate the auto-correlation function for the density $\rho(t)$ measured at some point $\mathbf{r}_0$ as a function of time: $K(\tau) = \langle \rho(t+\tau)\rho(t)\rangle$ where the angular brackets stand for time averaging of the (mass) density $\rho(t)$ and then carry out the Fourier transform $K(\tau)$,

$$K_\omega = \int_{-\infty}^{\infty} K(\tau) e^{i\omega\tau} d\tau.$$

Here the density distribution is assumed to be homogeneous so that K_ω does not depend on $\mathbf{r}_0$. It is worth noting that defined in such a way, K_ω coincides, up to

constant factor, with the energy spectrum in the frequency domain:

$$E_\omega = \frac{c_s^2}{\rho_0} K_\omega.$$

In the weak-turbulence approximation, in an isotropic case E_ω can be expressed in terms of the spectrum $\varepsilon(k)$ by means of the relation

$$E_\omega = \frac{4\pi\omega^2}{c_s^3}\,\varepsilon(k) \tag{4}$$

where $k = \omega/c_s$. This formula is the result of integration of the energy spectral density in the $k-\omega$ representation $E_{k\omega} = \varepsilon(k)\,\delta(\omega-\omega_k)$ with respect to $\mathbf{k}$. In the weak turbulence regime the spectral density $E_{k\omega}$ has the δ-function dependence upon frequency indicating that the wave ensemble is weakly nonlinear. In the presence of shocks, i.e. for the strong turbulence regime, such relations are not valid any more.

Our aim now is to determine the contribution of shocks associated with the density jumps to the E_ω spectrum. To achieve this one should take into account that at the instant t_i of jump passage through the measurement point $\mathbf{r}_0$ the first derivative $\partial\rho/\partial t$ is proportional to $\delta(t-t_i)$, i.e.,

$$\frac{\partial \rho}{\partial t} = \sum_i \Delta\rho_i \delta(t-t_i) + \text{regular terms.} \tag{5}$$

Assuming that density jumps $\Delta\rho_i$ and crossing times t_i are random quantities one can calculate the contribution of these singularities in Eq. (5) to the spectrum. The Fourier transform of the contribution associated with these terms can be written as follows:

$$\rho_\omega = \frac{i}{2\pi\omega}\sum_i \Delta\rho_i e^{-i\omega t_i}. \tag{6}$$

Here,

$$\rho_\omega = \int_{-\infty}^{\infty} \rho(t) e^{i\omega t} dt, \quad \rho(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\omega t}\rho_\omega d\omega.$$

To find the E_ω spectrum, one should square the absolute value of (6) and average the resulting expression. The averaging over crossing times t_i yields

$$E_\omega = \frac{c_s^2}{2\pi\rho_0\tau}\langle|\rho_\omega|^2\rangle = \frac{c_s^2\nu}{2\pi\rho_0\omega^2}\overline{(\Delta\rho)^2}, \tag{7}$$

where $\nu = N/\tau$ is the jump crossings frequency, here N is the number of discontinuities met during the averaging time τ, and $\overline{(\Delta\rho)^2}$ is the average value of $(\Delta\rho)^2$.

The very same approach can be applied for finding the spatial spectrum of acoustic turbulence for the one dimensional case ($D = 1$) when instead of (5) we have

$$\frac{\partial \rho}{\partial x} = \sum_i \Delta\rho_i \delta(x-x_i) + \text{regular terms} \tag{8}$$

where x_i are the positions of jumps along x-axis. Hence we get the following $1D$ spectrum $E(k)$:

$$E_1(k) = \frac{n_1 c_s^2}{2\pi\rho_0 k^2}\,\overline{(\Delta\rho)^2}. \tag{9}$$

Here, n_1 is the number density of shocks per unit length, ρ_0 is the mean density of medium (per unit length), and $\overline{(\Delta\rho)^2}$ is the mean-square of the density jump at the discontinuity.

This calculation can be considered in isotropic three dimensional case as the estimate of the correlation function along any chosen straight line. Now we shall make use of it to obtain the three-dimensional (3D) isotropic spectrum (9). To this end one should notice that the density correlation function for isotropic turbulence, $\phi(y_1) = \langle\rho(x_1+y_1,x_2,x_3)\rho(x_1,x_2,x_3)\rangle$, has the Fourier spectrum (with respect to only one single variable y_1!) that coincides, to within a factor, with Eq. (9):

$$\phi_k = \frac{N_1}{2\pi k^2}\,\overline{(\Delta\rho)^2}, \tag{10}$$

where N_1 in this case should be considered as the mean linear density of discontinuities. This correlation function ϕ_k is related to the three-dimensional Fourier spectrum

$$\Phi(|\mathbf{k}|) = \int \phi(\mathbf{r})e^{-i\mathbf{k}\cdot\mathbf{r}}d\mathbf{r}$$

by the following formula

$$\phi_{k_1} = \int \Phi(|\mathbf{k}|)d\mathbf{k}_\perp = \pi\int_{k_1^2}^{\infty}\Phi(s)ds^2$$

where $\mathbf{k}_\perp$ is the component of the wave vector $\mathbf{k}$ perpendicular to the x-axis. This allows one to obtain differentiating this equality with respect to k_1 the following relation:

$$\Phi(k) = -\frac{1}{2\pi k}\frac{d\phi_k}{dk}.$$

Then, substituting Eq. (10), one can find the following expression for the spectrum $E_3(k)$:

$$E_3(k) = \frac{2N_1 c_s^2}{\pi\rho_0 k^2}\,\overline{\Delta\rho^2}. \tag{11}$$

The same approach is applicable in two-dimensional case. It is a little more difficult technically because it requires to solve the integral Abel equation (e.g. compare with [12, 13]). However, making calculations one can find that the spectrum dependence upon k has the same spectral index as in one- and three- dimensional cases:

$$E_2(k) \propto k^{-2}.$$

According to our knowledge spectrum (9) was first obtained by Burgers in [22] and, its generalization for multi-dimensional situation was found (11) by Kadomtsev and Petviashvili [15]. In the next section we show how to generalize it taking into account the effect of anisotropy.

3 Anisotropic KP spectra

The analysis of anisotropic situation consists in taking account of two geometric factors. One is connected with the anisotropy of the caustics orientations, another with the anisotropy due to the "emission" diagram of each caustic that can be considered as the source of the wave spectrum. For the sake of simplicity and without loss of generality we can consider that each single caustic has the form of a disk. This simplification is based on important property of these objects, namely, during their evolution most of time their relative size in the perpendicular direction is much larger than along the parallel axis.

Let us consider a single caustic with radius R_i perpendicular to the x -axis and centered at the point $\mathbf{r}_0 = (x_0, \mathbf{r}_{\perp 0})$. To consider this effect equation (8) should be replaced by the following one:

$$\frac{\partial \rho}{\partial x} = \Delta\rho(|\mathbf{r}_\perp - \mathbf{r}_{\perp 0}|)\delta(x - x_0) + \text{regular} \quad \text{terms}. \tag{12}$$

Here, $\Delta\rho(r_\perp)$ is considered as a continuous cylindrically-symmetric function of $r_\perp$ that vanishes at the disk boundary $r_\perp = R$ ($\Delta\rho(R) = 0$) and remains zero outside the disk.

Then, the Fourier transform of the terms corresponding to singular part of Eq. (12) is given by the integral

$$\rho_{\mathbf{k}} = -\frac{i}{k_x} e^{-i\mathbf{k}\mathbf{r}_0} \int_{r_\perp \le R} \Delta\rho(r_\perp) e^{-i\mathbf{k}_\perp \mathbf{r}_\perp} d\mathbf{r}_\perp = -\frac{2\pi i}{k_x} e^{-i\mathbf{k}\mathbf{r}_0} \int_0^R r_\perp \Delta\rho(r_\perp) J_0(k_\perp r_\perp) dr_\perp,$$

where $\mathbf{k} = (k_x, \mathbf{k}_\perp)$ and $J_0(k_\perp r_\perp)$ is the Bessel function. This is the contribution from one single singularity. The total contribution from all discontinuities can be found as the sum:

$$\rho_{\mathbf{k}} = -2\pi i \sum_\alpha \frac{e^{-i\mathbf{k}\mathbf{r}_\alpha}}{\mathbf{k}\mathbf{n}_\alpha} \int_0^{R_\alpha} \Delta\rho(r_\perp)\ r_\perp J_0(k_{\perp\alpha} r_\perp) dr_\perp.$$

Here, $\mathbf{n}_\alpha$ is the normal unit vector to the discontinuity α, $\mathbf{r}_\alpha$ are the disk center coordinates and $k_{\perp\alpha}$ is the transverse projection of the wave vector $\mathbf{k}$ to the disc plane ($k_{\perp\alpha}^2 = k^2 - (\mathbf{k}\mathbf{n}_\alpha)^2$). It is worth noting here that the anisotropic characteristics of the spectrum is related to the anisotropy of the distribution of unit vectors $\mathbf{n}_\alpha$.

To find the spectrum of turbulence, one should average $|\rho_k|^2$ over all random variables. It is natural to assume that the coordinates $\mathbf{r}_\alpha$ of centers of caustics are distributed uniformly, the averaging over these variables results in

$$\overline{|\rho_k|^2} = 4\pi^2 N \left\langle \left| \frac{1}{k_\parallel} \int_0^R \Delta\rho(r_\perp) r_\perp J_0(k_\perp r_\perp) dr_\perp \right|^2 \right\rangle. \tag{13}$$

Here, $k_\parallel \equiv \mathbf{k}\mathbf{n}$, N is the density of discontinuities and the angular brackets stand for the averaging over characteristic sizes R and angles.

Our major interest here is in evaluation of the short-wavelength asymptotic behavior of the spectrum found (13), i.e., $k\overline{R} \gg 1$, where $\overline{R}$ is the characteristic disc radius. Thus, if $k_\perp \overline{R} \gg 1$, then the expression inside the integral in Eq. (13) represents rapidly oscillating function (due to the Bessel function $J_0(k_\perp r_\perp)$) and therefore the integral can be evaluated by means of the stationary phase method. Using the relation

$$xJ_0(\alpha x) = \frac{1}{\alpha}\frac{d}{dx}\left(xJ_1(\alpha x)\right),$$

and integrating by parts we have

$$\int_0^R \Delta\rho(r_\perp) r_\perp J_0(k_\perp r_\perp) dr_\perp = -\frac{R}{k_\perp}\int_0^1 J_1(k_\perp Rx)\; x\frac{d}{dx}\Delta\rho(Rx)dx$$

where the property $\Delta\rho(R) = 0$ is used. Having in mind that the major input to the integral comes from the vicinity of the boundary at $r_\perp = R$ we can use the asymptotic expression for the Bessel function at large $k_\perp R$

$$J_1(z) \simeq \sqrt{\frac{2}{\pi z}}\cos\left(z - \frac{3\pi}{4}\right),$$

and obtain the following asymptotic estimate of the integral:

$$-\sqrt{\frac{2R}{\pi}}\frac{1}{k_\perp^{3/2}}\int_0^1 \cos\left(k_\perp Rx\right)\; x^{1/2}\frac{d}{dx}\Delta\rho(Rx)dx \sim -\sqrt{\frac{2R}{\pi}}\frac{\sin\left(k_\perp R\right)}{k_\perp^{5/2}}\;\frac{d}{dr_\perp}\Delta\rho(r_\perp)\Big|_{r_\perp=R}.$$

where the main contribution to the integral comes explicitly from the boundary $r_\perp = R$.

Hence the resulting spectrum $\tilde{\epsilon}(\mathbf{k})$ (still prior to angular averaging!) can be written as follows

$$\tilde{\epsilon}_1(\mathbf{k}) = \frac{4\pi n c_s^2}{\rho_0}\frac{\langle\Psi\rangle}{k_\perp^5 k_\parallel^2},\quad \Psi = R\left(\frac{d}{dr_\perp}\Delta\rho(r_\perp)\Big|_{r_\perp=R}\right)^2. \tag{14}$$

It is worth noting that the stationary phase method used for the evaluation of the integral in Eq. (13) is applicable at almost all angles θ (θ is the angle between the vectors $\mathbf{k}$ and $\mathbf{n}$) but not in two narrow cones $\theta \le \vartheta_0 = \left(k\overline{R}\right)^{-1}$ and $\pi - \theta \le \vartheta_0$. Inside these cones one can find that the integral can be considered to be independent on k (suggesting $\cos\left(k_\perp Rx\right) \approx 1$). The spectrum $\tilde{\epsilon}(\mathbf{k})$ inside these cones is then given by

$$\tilde{\epsilon}_2(\mathbf{k}) \approx \frac{4\pi n c_s^2}{\rho_0}\frac{\langle\Gamma^2\rangle}{k_\parallel^2}, \tag{15}$$

where

$$\Gamma = \int_0^R \Delta\rho(r_\perp) r_\perp dr_\perp.$$

From Eq. (14) one can see that the spectrum $\tilde{\epsilon}_1(\mathbf{k})$ contains three singularities, namely, at angles θ close to $0, \pi$ and $\pi/2$. For angles close to the cone $\theta \approx \left(k\overline{R}\right)^{-1}$

and $\pi - \theta \approx \left(k\overline{R}\right)^{-1}$ expression (14) matches Eq.(15). For angles close to $\pi/2$, in Eq. (14) an additional effect should be taken into account, namely, the bending of the caustics. If a is the characteristic value of bending, Eq. (14) is valid in the region $|\theta - \pi/2| > (ka)^{-1}$.

Distributions (14) and (15) allow one to carry out the averaging over angular distribution of caustics normals, if their distribution is anisotropic, i.e., calculate the angular dependence of the energy spectrum $E(\mathbf{k}) = k^2\overline{\tilde{\epsilon}(\mathbf{k})}$ in short enough scales $\left(k\overline{R}\right) \gg 1$.

First, let us check that in the isotropic case the KP spectrum (11) follows from the above calculations. In this case averaging over the angles corresponds to the integration of Eqs. (14) and (15) with respect to θ. The integration of expression (15) near poles $\theta = 0, \pi$ gives

$$E_2(k) = 4\pi k^2 \int_0^{\theta_0} \tilde{\epsilon}_2(k)\theta d\theta = \frac{16\pi^2 n c_s^2}{\rho_0 k^2} \frac{\langle\Gamma^2\rangle}{\overline{R}^2}. \tag{16}$$

In the integration of Eq. (14) over angles, the main contribution to the spectrum comes from the angles close to $0, \pi$ and $\pi/2$, where the spectrum (14) has singularities. For $\theta \to 0$ (for $\theta \to \pi$), the integration is cut off at angles $\theta_k = \vartheta_0$ (at $\theta_k = \pi - \vartheta_0$), and for $\theta \to \pi/2$ it is cut off at angles $|\pi/2 \pm \theta_k| \approx (ka)^{-1}$. As the result of averaging of the expression (14) we get that the main contributions coming from the angles near the cone:

$$E_1(k) \approx \frac{16\pi^2 n c_s^2}{\rho_0 k^2} \frac{\overline{R}^3 \langle\Psi\rangle}{3}. \tag{17}$$

The spectrum $E(k)$ is given by the sum of (16) and (17),

$$E(k) \approx \frac{16\pi^2 n c_s^2}{\rho_0 k^2} \left(\frac{\langle\Gamma^2\rangle}{\overline{R}^2} + \frac{\overline{R}^3 \langle\Psi\rangle}{3} \right), \tag{18}$$

that has the same dependence on k as the KP spectrum (11) obtained in a little different way from conventional consideration.

Our above performed analysis of spectral angular dependence upon the angle that is applicable first of all to one single caustic leads to the conclusion that the "emission diagram" of it represents narrow cones around $\theta = 0, \frac{\pi}{2}, \pi$. The angular width of these cones is proportional to $\left(k\overline{R}\right)^{-1}$ around $\theta = 0, \pi$ and to $(ka)^{-1}$ around $\theta = \pi/2$. In the case of anisotropic distribution of caustics normals the angular dependence of the spectrum will simply reproduce this angular dependence if the width $\Delta\theta$ of this angular distribution is sufficiently larger than the width of these cones. This can be resumed as follows, if the angular distribution of caustics normals is wide enough so that $\Delta\theta > \vartheta_0$ then after averaging of $\tilde{\epsilon}(\mathbf{k})$ one should get the same dependence of the spectrum upon k at large enough k as for the isotropic KP spectrum, i.e. $\sim k^{-2}$ with the degree of the anisotropy exactly the same as the distribution of caustics normals. From the other hand if the angular distribution of

caustics normals is sufficiently narrow , e.g., if all falling shock fronts are oriented almost unidirectionally (this can be caused, e.g., by pumping or initial/boundary conditions), then the spectrum will have a sharp peak in this direction. If the width $\Delta\theta$ of angular distribution is narrower than ϑ_0, i.e., if $\Delta\theta < \vartheta_0$, then the spectrum $E(k,\theta)$ up to the multiplier k^2 will repeat the distribution given by Eqs. (14) and (15), namely, the distribution in k-space will have the form of the *jet* . In the cone $\theta < \vartheta_0$ the spectrum has a maximum with fall off at large k in accordance with (14), i.e $\sim k_{\parallel}^{-2}$ like the KP spectrum (11). At larger angles, $\theta > \vartheta_0$ the spectrum $E(k,\theta)$ will rapidly decrease in the transverse direction proportionally to $k_{\perp}^{-5}$.

Note, however, that this asymptotic behavior is intermediate, because $\vartheta_0 = (k\overline{R})^{-1}$ decreases with increasing k. For this reason, when averaging over angles, singularities in (14) become essential for $\theta \to 0$, and, starting from certain $k = k^*$, the spectrum will decrease as k^{-2} with increasing k. The angular width of the spectrum will be of order $\Delta\theta$.

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